Rigid affine surfaces with isomorphic A2-cylinders

Adrien Dubouloz

RIGID AFFINE SURFACES WITH ISOMORPHIC $\mathbb{A}^2$-CYLINDERS

ADRIEN DUBOULOZ

Abstract. We construct families of smooth affine surfaces with pairwise non isomorphic $\mathbb{A}^1$-cylinders but whose $\mathbb{A}^2$-cylinders are all isomorphic. These arise as complements of cuspidal hyperplane sections of smooth projective cubic surfaces.

Introduction

The Zariski Cancellation Problem, which asks whether two, say smooth affine, algebraic varieties X and Y with isomorphic cylinders $X \times \mathbb{A}^n$ and $Y \times \mathbb{A}^n$ for some $n \geq 1$ are isomorphic themselves, has been studied very actively during the past decades culminating recently with a negative solution in dimension 3 and positive characteristic for the case $X = \mathbb{A}^3$ [7]. The situation in the complex case, and more generally over any algebraically closed field of characteristic zero, is more contrasted: cancellation is known to hold for curves [1] and for $\mathbb{A}^2$ [6], but many counter-examples in every dimension higher or equal to 2 have been discovered (see [14] for a survey), inspired by the two pioneering constructions of Hochster [8] and Danielewski [2].

Essentially all known families are counter-examples to the cancellation of 1-dimensional cylinders which arise from the existence of nontrivial decompositions of certain locally trivial $\mathbb{A}^2$-bundles over a base scheme Z. Namely, Hochster type constructions rely on the existence of non free, 1-stably free, projective modules which in geometric term correspond to non trivial decompositions of the trivial bundle $Z \times \mathbb{A}^{r+1} \to Z$ into a trivial $\mathbb{A}^1$-bundle over a nontrivial vector bundle $E \to Z$ of rank $r \geq 1$. For every such bundle, the varieties $X = E$ and $Y = Z \times \mathbb{A}^r$ have isomorphic cylinders $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$, and one then gets a counter-example to the cancellation problem whenever X and Y, which by definition are non isomorphic as schemes over Z, are actually non isomorphic as abstract algebraic varieties [10]. In contrast, Danielewski type constructions usually involve non trivial decompositions of a principal homogeneous $\mathbb{G}_a^2$-bundle $W \to Z$ into pairs $W \to X \to Z$ and $W \to Y \to Z$ consisting of trivial $\mathbb{G}_a$-bundles over nontrivial $\mathbb{G}_a$-bundles $X \to Z$ and $Y \to Z$ with affine total spaces, with the property that W is isomorphic to the fiber product $W = X \times_Z Y$. The isomorphism $X \times \mathbb{A}^1 \simeq W \simeq Y \times \mathbb{A}^1$ is granted by definition, and similarly as in the previous type of construction, one obtains counter-examples to the cancellation problem as soon as X and Y are not isomorphic as abstract varieties.

Non-cancellation phenomena with respect to higher dimensional cylinders are more mysterious. In fact, it seems for instance that not even a single explicit example of a pair of non-isomorphic varieties X and Y which fail the $\mathbb{A}^2$-cancellation property in a minimal way, in the sense that $X \times \mathbb{A}^2$ and $Y \times \mathbb{A}^2$ are isomorphic while $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are still non isomorphic, is known so far. In this article, we fill this gap by constructing a positive dimensional moduli of smooth affine surfaces which fail the $\mathbb{A}^2$-cancellation property minimally. That such varieties exist was certainly a natural expectation, and their existence is therefore neither really surprising, nor probably exciting in itself due to the abundance of simpler counter-examples to the cancellation problem. Their interest lies rather in the fact that they provide additional insight on the algebro-geometric properties that a variety should satisfy in order to fail cancellation.

Indeed, it follows from Iitaka-Fujita strong Cancellation Theorem [9] that a smooth affine variety X which fails cancellation has negative logarithmic Kodaira dimension, a property conjecturally equivalent in dimension higher or equal to 3 to the fact that X is covered by images of the affine line and equivalent for surfaces to the existence of an $\mathbb{A}^1$-fibration $\pi : X \to C$ over a smooth curve [12], i.e. a flat fibration with general fibers isomorphic to the affine line. In the particular case of the cancellation problem for 1-dimensional cylinders, a further striking discovery of Makar-Limanov is that the existence of nontrivial actions of the additive group $\mathbb{G}_a$ on X is a necessary condition for non-cancellation. Namely, Makar-Limanov semi-rigidity theorem [11] (see also [5, Proposition 9.23]) asserts that if X is rigid, i.e. does not admit any nontrivial $\mathbb{G}_a$-action, then the projection $\mathrm{pr}_X : X \times \mathbb{A}^1 \to X$ is invariant under all $\mathbb{G}_a$-actions on X. As a consequence, if either X or Y is rigid then every isomorphism between $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ descends to an isomorphism between X and Y. Combined with Fieseler's topological

2010 *Mathematics Subject Classification.* 14R05, 14R25, 14E07.

Key words and phrases. Rigid surfaces, Cancellation Problem.

This research was partialy supported by ANR Grant "BirPol" ANR-11-JS01-004-01.

description of algebraic quotient morphisms of $\mathbb{G}_a$-actions on smooth complex affine surfaces [4], these results imply that a smooth affine surface which fails $\mathbb{A}^1$-cancellation must admit a nontrivial $\mathbb{G}_a$-action whose algebraic quotient morphism $\pi : X \to X//\mathbb{G}_a = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_a})$ is not a locally trivial $\mathbb{A}^1$-bundle. This holds of course for the two smooth surfaces $xz - y(y+1) = 0$ and $x^2z - y(y+1) = 0$ used by Danielewski in his celebrated counter-example, showing a posteriori that his construction was essentially optimal in this dimension. The rich families of existing counter-examples to $\mathbb{A}^1$-cancellation in dimension 2 lend support to the conjecture that every smooth affine surface which is neither rigid nor isomorphic to the total space of line bundle over an affine curve fails the $\mathbb{A}^1$-cancellation property.

In view of this conjecture, a smooth affine surface X which fails the $\mathbb{A}^2$-cancellation property in a minimal way must be simultaneously rigid and equipped with an $\mathbb{A}^1$-fibration $\pi : X \to C$ over a smooth curve, and the well known fact that $\mathbb{A}^1$-fibrations over affine curves are algebraic quotient morphisms of nontrivial $\mathbb{G}_a$-actions implies further that C must be projective. This is precisely the case for the family of surfaces we construct in this article, a particular example being the smooth affine cubic surfaces

$$X = \{(-1+\alpha\sqrt[3]{2}x_2+\alpha\sqrt[3]{2}x_3)^3+8(x_1^3+x_2^3+x_3^3) = 0\} \text{ and } X' = \{(-1+2\alpha x_1+\alpha\sqrt[3]{2}x_2+\alpha\sqrt[3]{2}x_3)^3+8(x_1^3+x_2^3+x_3^3) = 0\}$$

in $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3])$, where $\alpha = \exp(i\pi/3)$, which are both rigid and equipped with an $\mathbb{A}^1$-fibration over $\mathbb{P}^1$. These arise as the complements in the Fermat cubic surface $V = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\}$ in $\mathbb{P}^3$ of the plane cuspidal cubics $C = \{-(x_2 + x_3)^3 + 4(x_1^3 + x_2^3 + x_3^3)\} = 0$ and $C' = \{-((\sqrt[3]{2})^2 x_1 + x_2 + x_3)^3 + 4(x_1^3 + x_2^3 + x_3^3) = 0\}$ obtained by intersecting V with its tangent hyperplane at the points $p = [\alpha\sqrt[3]{2} : 0 : 1 : 1]$ and $p' = [\alpha\sqrt[3]{2} : \sqrt[3]{2} : -1 : 1]$ respectively. The group of automorphisms of V being isomorphic to $\mathbb{Z}_3 \times \mathfrak{S}_4$, where $\mathbb{Z}_3$ is the 3-torsion subgroup of $\mathrm{PGL}(4;\mathbb{C})$ and where $\mathfrak{S}_4$ denotes the group of permutations of the variables, the fact that p and p' do not belong to a same $\mathrm{Aut}(V)$-orbit implies that the pairs (V, C) and (V, C') are not isomorphic. Our main result just below then implies in turn that X and X' are non isomorphic, with isomorphic $\mathbb{A}^2$-cylinders $X \times \mathbb{A}^2$ and $X' \times \mathbb{A}^2$.

Theorem. *Let (V_i, C_i), $i = 1, 2$, be non isomorphic pairs consisting of a smooth cubic surface $V_i \subset \mathbb{P}^3$ and a cuspidal hyperplane section $C_i = V_i \cap H_i$. Then the affine surfaces $X_i = V_i \setminus C_i$ are non isomorphic, with non isomorphic $\mathbb{A}^1$-cylinders $X_i \times \mathbb{A}^1$ but with isomorphic $\mathbb{A}^2$-cylinders $X_i \times \mathbb{A}^2$, $i = 1, 2$.*

As a consequence, all smooth affine surfaces arising as complements of cuspidal hyperplane sections of smooth projective cubic surfaces have isomorphic $\mathbb{A}^2$-cylinders. Noting that the projective closure in $\mathbb{P}^3$ of the surface $X_0 \subset \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ with equation $x^2y + y^2 + z^3 + 1 = 0$ is a smooth cubic surface intersecting the plane at infinity along the cuspidal cubic $x^2y + z^3 = 0$, we obtain the following:

Corollary. *Let X be a smooth affine surface isomorphic to the complement of a cuspidal hyperplane section of a smooth projective cubic surface. Then $X \times \mathbb{A}^2$ is isomorphic to the affine cubic fourfold $Z \subset \mathbb{A}^5 = \mathrm{Spec}(\mathbb{C}[x, y, z][u, v])$ with equation $x^2y + y^2 + z^3 + 1 = 0$. Furthermore, X is isomorphic to the geometric quotient of a proper action of the group $\mathbb{G}_a^2$ on Z.*

The scheme of the proof of the Theorem given in the next section is the following. The fact that the affine surfaces X_1 and X_2 are non-isomorphic follows from the non-isomorphy of the pairs (V_1, C_1) and (V_2, C_2) via an argument of classical birational geometry of projective cubic surfaces, which simultaneously renders the conclusion that X_1 and X_2 are rigid. The non isomorphy of the cylinders $X_1 \times \mathbb{A}^1$ and $X_2 \times \mathbb{A}^1$ is then a straightforward consequence Makar-Limanov's semi-rigidity Theorem.

The existence of an isomorphism between the $\mathbb{A}^2$-cylinders $X_1 \times \mathbb{A}^2$ and $X_2 \times \mathbb{A}^2$ is derived in two steps: the first one consists of another instance of a Danielewski fiber product trick argument, which provides a smooth affine threefold W equipped with simultaneous structures of line bundles $\pi_1 : W \to X_1$ and $\pi_2 : W \to X_2$ over X_1 and X_2. But here, in contrast with the situation in Danielewski's counter-example, the fact that the $\mathbb{A}^1$-cylinders over X_1 and X_2 are not isomorphic implies that these two line bundles cannot be simultaneously trivial. Nevertheless, the crucial observation which enables a second step, reminiscent to Hochster construction, is that the pull-backs via the isomorphisms $\pi_i^* : \mathrm{Pic}(X_i) \to \mathrm{Pic}(W)$ of the classes of these lines bundles in the Picard groups of X_1 and X_2, say L_1 and L_2, coincide. Letting $q : E \to W$ be a line bundle representing the common inverse in $\mathrm{Pic}(W)$ of $\pi_1^* L_1 = \pi_2^* L_2$, the composition $\pi_i \circ q : E \to X_i$ is then a vector bundle of rank 2 isomorphic to the direct sum $L_i \oplus L_i^\vee$, where $L_i^\vee$ denotes the dual of L_i, hence isomorphic to $\det(E) \oplus \mathbb{A}^1_{X_i} = (L_i \otimes L_i^\vee) \oplus \mathbb{A}^1_{X_i} \simeq \mathbb{A}^1_{X_i} \oplus \mathbb{A}^1_{X_i}$ by virtue of result of Pavaman Murthy [13] asserting that every vector bundle on a smooth affine surface birationaly equivalent to a ruled surface is isomorphic to the direct sum of a trivial bundle with a line bundle.

$$\begin{tikzcd} X_1 \times \mathbb{A}^2 \arrow[r, "\sim"] \arrow[dd, "\mathrm{pr}_{S_1}"'] & E \arrow[d, "q"'] & X_2 \times \mathbb{A}^2 \arrow[l, "\sim"'] \arrow[dd, "\mathrm{pr}_{X_2}"] \\ & W \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2. \end{tikzcd}$$

The construction of these isomorphisms suggests the following strengthening of the above conjecture characterizing smooth affine surfaces failing the $\mathbb{A}^1$-cancellation property, which would settle the question of the behavior of smooth affine surfaces under stabilization by affine spaces:

Conjecture. *A smooth affine surface X with negative logarithmic Kodaira dimension is either isomorphic to the total space of a line bundle over a curve, or it fails the $\mathbb{A}^2$-cancellation property. Furthermore, every non rigid X which fails the $\mathbb{A}^2$-cancellation property also fails the $\mathbb{A}^1$-cancellation property.*

1. Proof of the theorem

1.1. **Rigid affine cubic surfaces.** Given a smooth cubic surface $V \subset \mathbb{P}^3$ and a hyperplane section $V \cap H$ consisting of an irreducible plane cuspidal cubic C, the restriction of the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from the singular point p of C induces a rational map $V \dashrightarrow \mathbb{P}^2$ of degree 2 with p as a unique proper base point. Its lift to the blow-up $\alpha : Y \to V$ of V at p coincides with the morphism $\theta : Y \to \mathbb{P}^2$ defined by the anti-canonical linear system $|-K_Y|$ and it factors through a birational morphism $Y \to Z$ to the anti-canonical model $Z = \mathrm{Proj}_{\mathbb{C}}(\bigoplus_{m\geq 0} H^0(Y, -mK_Y))$ of Y, followed by a Galois double cover $Z \to \mathbb{P}^2$ ramified over a quartic curve. The nontrivial involution of the double cover $Z \to \mathbb{P}^2$ lifts to a biregular involution of Y exchanging the proper transform of C and the exceptional divisor E of α. This involution descends back to a birational map $G_p : V \dashrightarrow V$, called the *Geiser involution of V with center at p*, which contracts C to p and restricts to a biregular involution $j_p : X \to X$ of the affine complement X of C in V.

Lemma 1. *Let X_i be the complements of cuspidal hyperplanes sections $C_i = V_i \cap H_i$ with respective singular points p_i of smooth cubic surfaces $V_i \subset \mathbb{P}^3$, $i = 1, 2$. Then for every isomorphism $\psi : X_1 \xrightarrow{\sim} X_2$, the birational map $\overline{\psi} : V_1 \dashrightarrow V_2$ extending ψ is either an isomorphism of pairs $(V_1, C_1) \xrightarrow{\sim} (V_2, C_2)$ or it factors in a unique way as the composition of the Geiser involution $G_{p_1} : V_1 \dashrightarrow V_1$ followed by an isomorphism of pairs $(V_1, C_1) \xrightarrow{\sim} (V_2, C_2)$. In particular, X_1 and X_2 are isomorphic if and only if so are the pairs (V_1, C_1) and (V_2, C_2).*

Proof. Letting $\alpha_i : Y_i \to V_i$ be the blow-up of V_i at p_i, with exceptional divisor E_i, X_i is isomorphic to $Y_i \setminus (C_i \cup E_i)$ where we identified C_i and its proper transform in Y_i. The birational map $\overline{\psi} : V_1 \dashrightarrow V_2$ lifts to a birational $\overline{\Psi} = \alpha_2^{-1} \circ \overline{\psi} \circ \alpha_1 : Y_1 \dashrightarrow Y_2$ extending ψ, and the assertion is equivalent to the fact that $\overline{\Psi}$ is an isomorphism of pairs $(Y_1, C_1 \cup E_1) \xrightarrow{\sim} (Y_2, C_2 \cup E_2)$. Since Y_1 and Y_2 are smooth with the same Picard rank $\rho(Y_i) = 8$, this holds provided that either $\overline{\Psi}$ or $\overline{\Psi}^{-1}$ is a morphism. So suppose for contradiction that $\overline{\Psi}$ or $\overline{\Psi}^{-1}$ are both strictly birational and let $Y_1 \xleftarrow{\sigma_1} W \xrightarrow{\sigma_2} Y_2$ be the minimal resolution of $\overline{\Psi}$. Since Y_1 and Y_2 are smooth and $\overline{\Psi}$ and $\overline{\Psi}^{-1}$ are both strictly birational, σ_1 consists of a non-empty sequence of blow-ups of smooth points whose centers lie over $C_1 \cup E_1$, while σ_2 is a non-empty sequence of contractions of successives (-1)-curves on W supported on the total transform $\sigma_1^{-1}(C_1 \cup E_1)$ of $C_1 \cup E_1$. Furthermore, the minimality assumption implies that the first curve contracted by σ_2 is the proper transform in W of C_1 or E_1. Since C_1 and E_1 are (-1)-curves in Y_1, the only possibility is thus that all successive centers of σ_1 lie over $E_1 \setminus C_1$ (resp. $C_1 \setminus E_1$) and that the first curve contracted by σ_2 is the proper transform of E_1 (resp. C_1). But since C_1 and E_1 are tangent in Y_1, so are their proper transforms in W, and then the image of C_1 (resp. E_1) by the contraction $\tau : W \to W'$ of E_1 (resp. C_1) factoring $\sigma_2 : W \to Y_2$ would be singular. Since it cannot be contracted at any further step, its image by σ_2 would be a singular curve contained in $Y_2 \setminus X_2 = C_2 \cup E_2$, which is absurd. $\square$

Corollary 2. *Let X be the complement of a cuspidal hyperplane section C of a smooth cubic surface $V \subset \mathbb{P}^3$. Then there exists a split exact sequence*

$$0 \to \mathrm{Aut}(V, C) \to \mathrm{Aut}(X) \to \{\mathrm{id}_X, j_p\} \simeq \mathbb{Z}_2 \to 0,$$

where $\mathrm{Aut}(V,C)$ *is the automorphism group of the pair* (V,C) *and* $j_p : X \overset{\sim}{\to} X$ *is the biregular involution induced by the Geiser involution of* V *with center at the singular point* p *of* C*. In particular,* $\mathrm{Aut}(X)$ *is a finite group, isomorphic to* $\mathbb{Z}_2$ *for a general smooth cubic surface* V*.*

Proof. We view $\mathrm{Aut}(V,C)$ as a subgroup of $\mathrm{Aut}(X)$ via the homomorphism which associates to every automorphism of V preserving C, hence X, its restriction to X. Since by virtue of the previous lemma, the extension of every automorphism φ of X to a birational self-map $\overline{\varphi} : V \dashrightarrow V$ is either an automorphism of the pair (V,C) or the composition of the Geiser involution $G_p : V \dashrightarrow V$ with an automorphism of this pair, the first assertion follows. The second assertion is a consequence of the fact that the automorphism group $\mathrm{Aut}(V)$ of a smooth cubic surface V is always finite, actually trivial for a general such surface. ☐

The following proposition provides the first part of the proof of the theorem:

Proposition 3. *Let* X_i *be the complements of cuspidal hyperplanes sections* $C_i = V_i \cap H_i$ *of smooth cubic surfaces* $V_i \subset \mathbb{P}^3$*,* $i = 1,2$*. If the pairs* (V_1,C_1) *and* (V_2,C_2) *are not isomorphic then the* $\mathbb{A}^1$*-cylinders* $X_1 \times \mathbb{A}^1$ *and* $X_2 \times \mathbb{A}^1$ *are not isomorphic.*

Proof. The rigidity of X_i asserted by Corollary 2 implies by virtue of [5, Proposition 9.23] that the Makar-Limanov invariant $\mathrm{ML}(X_i \times \mathbb{A}^1)$ of $X_i \times \mathbb{A}^1$ is equal to the sub-algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ of $\Gamma(X_i, \mathcal{O}_{X_i})[t] = \Gamma(X_i \times \mathbb{A}^1, \mathcal{O}_{X_i \times \mathbb{A}^1})$. Since every isomorphism between two algebras induces an isomorphism between their Makar-Limanov invariants, it follows that every isomorphism $X_1 \times \mathbb{A}^1 \overset{\sim}{\to} X_2 \times \mathbb{A}^1$ descends to a unique isomorphism $\psi : X_1 \overset{\sim}{\to} X_2$ making the following diagram commutative

$$\begin{tikzcd} X_1 \times \mathbb{A}^1 \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_{X_1}"'] & X_2 \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_{X_2}"] \\ X_1 \arrow[r, "\psi"] & X_2. \end{tikzcd}$$

On the other hand, the hypothesis that the pairs (V_1,C_1) and (V_2,C_2) are not isomorphic combined with Lemma 1, implies that X_1 is not isomorphic to X_2 and so, $X_1 \times \mathbb{A}^1$ is not isomorphic to $X_2 \times \mathbb{A}^1$. ☐

1.2. **Isomorphisms between $\mathbb{A}^2$-cylinders.** As explained above, the first step of the construction is a Danielewski fiber product trick creating a smooth affine threefold W which is simultaneously the total space of a line bundle over X_1 and X_2. To setup such a fiber product argument, we first construct a certain smooth algebraic space $\delta : B \to \mathbb{P}^1$ with the property that every complement X of an irreducible cuspidal hyperplane section C of a smooth cubic surface $V \subset \mathbb{P}^3$ admits the structure of an étale locally trivial $\mathbb{A}^1$-bundle $\rho : X \to B$.

1.2.1. Letting $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{C}[z_0, z_1])$, the algebraic space $\delta : B \to \mathbb{P}^1$ is obtained by the following gluing procedure:

1) We let $U_\infty = \mathbb{P}^1 \setminus \{0\} \simeq \mathrm{Spec}(\mathbb{C}[w_\infty])$, where $w_\infty = z_1/z_0$, and we let $\delta_\infty : B_\infty \to U_\infty$ be the scheme isomorphic to affine line with a 6-fold origin obtained by gluing six copies $\delta_{\infty,i} : U_{\infty,i} \overset{\sim}{\to} U_\infty$, $i = 1,\ldots,6$ of U_∞, by the identity outside the points $\infty_i = \delta_{\infty,i}^{-1}(\infty)$.

2) We $U_0 = \mathbb{P}^1 \setminus \{\infty\} \simeq \mathrm{Spec}(\mathbb{C}[w_0])$, where $w_0 = z_0/z_1$, we let $\xi : \tilde{U}_0 \simeq \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[\tilde{w}_0]) \to U_0 \simeq \mathrm{Spec}(\mathbb{C}[w_0])$, $\tilde{w}_0 \mapsto w_0 = \tilde{w}_0^3$ be the triple Galois cover totally ramified over 0 and étale elsewhere, and we let $\tilde{\delta}_0 : \tilde{B}_0 \to \tilde{U}_0$ be the scheme isomorphic to the affine line with 3-fold origin obtained by gluing three copies $\tilde{U}_{0,1}$, $\tilde{U}_{0,\omega}$ and $\tilde{U}_{0,\omega^2}$ of $\tilde{U}_0$ by the identity outside their respective origins $\tilde{0}_{0,1}$, $\tilde{0}_{0,\omega}$ and $\tilde{0}_{0,\omega^2}$. The action of the Galois group $\mu_3 = \{1, \omega, \omega^2\}$ of complex third roots of unity of the covering ξ lifts to fixed point free action on $\tilde{B}_0$ given locally by $\tilde{U}_{0,\eta} \ni \tilde{z}_0 \mapsto \omega\tilde{z}_0 \in \tilde{U}_{0,\omega\eta}$. Since the latter has trivial isotropies, a geometric quotient exists in the category of algebraic spaces in the form an étale locally trivial μ_3-bundle $\tilde{B}_0 \to \tilde{B}_0/\mu_3 = B_0$ over a certain algebraic space B_0. Furthermore, the μ_3-equivariant morphism $\tilde{\delta}_0 : \tilde{B}_0 \to \tilde{U}_0$ descends to a morphism $\delta_0 : B_0 \to \tilde{U}_0 /\!/ \mu_3 \simeq U_0$ restricting to an isomorphism over $U_0 \setminus \{0\}$ and totally ramified over $\{0\}$, with ramification index 3.

3) Finally, $\delta : B \to \mathbb{P}^1$ is obtained by gluing $\delta_\infty : B_\infty \to U_\infty$ and $\delta_0 : B_0 \to U_0$ along the open sub-schemes $\delta_\infty^{-1}(U_0 \cap U_\infty) \simeq \mathrm{Spec}(\mathbb{C}[w_\infty^{\pm 1}])$ and $\delta_0^{-1}(U_0 \cap U_\infty) \simeq \mathrm{Spec}(\mathbb{C}[w_\infty^{\pm 1}])$ by the isomorphism $w_\infty \mapsto w_0 = w_\infty^{-1}$.

Remark 4. Letting p_0 be the unique closed point of B over $0 \in U_0 \subset \mathbb{P}^1$, we have $\delta^{-1}(0) = 3p_0$ while the restriction of δ over $U_0 \setminus \{0\}$ is an isomorphism. This implies that B is not a scheme, for otherwise the restriction of δ over U_0 would be an isomorphism by virtue of Zariski Main Theorem. In fact, p_0 is a point which does not admit any affine open neighborhood V: otherwise the inverse image of V by the finite morphism $\tilde{B}_0 \to \tilde{B}_0/\mu_3 = B_0$ would be an affine open sub-scheme of $\tilde{B}_0$ containing the three points $\tilde{0}_{0,1}$, $\tilde{0}_{0,\omega}$ and $\tilde{0}_{0,\omega^2}$ which is impossible.

1.2.2. Since the automorphism group of $\mathbb{A}^1$ is the affine group $\mathrm{Aff}_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$, every étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to B$ is an affine-linear bundle whose isomorphy class is determined by an element in the non-abelian cohomology group $H^1_{\text{ét}}(B, \mathrm{Aff}_1)$. Equivalently $\rho : S \to B$ is a principal homogeneous bundle under the action of a line bundle $L \to B$, considered as a locally constant group scheme over B for the group law induced by the addition of germs of sections, whose class in $\mathrm{Pic}(B) \simeq H^1_{\text{ét}}(B, \mathbb{G}_m)$ coincides with the image of the isomorphy class of $\rho : S \to B$ in $H^1_{\text{ét}}(B, \mathrm{Aff}_1)$ by the map $H^1_{\text{ét}}(B, \mathrm{Aff}_1) \to H^1_{\text{ét}}(B, \mathbb{G}_m)$ in the long exact sequence of non-abelian cohomology

$$0 \to H^0(B, \mathbb{G}_a) \to H^0(B, \mathrm{Aff}_1) \to H^0(B, \mathrm{Aff}_1) \to H^1_{\text{ét}}(B, \mathbb{G}_a) \to H^1_{\text{ét}}(B, \mathrm{Aff}_1) \to H^1_{\text{ét}}(B, \mathbb{G}_m)$$

associated to the short exact sequence $0 \to \mathbb{G}_a \to \mathrm{Aff}_1 \to \mathbb{G}_m \to 0$. Isomorphy classes of principal homogeneous under a given line bundle $L \to B$ are are in turn classified by the cohomology group $H^1_{\text{ét}}(B, L)$.

Proposition 5. *The complement X of a cuspidal hyperplane section C of a smooth cubic surface $V \subset \mathbb{P}^3$ admits an $\mathbb{A}^1$-fibration $f : X \to \mathbb{P}^1$ which factors through a principal homogeneous bundle $\rho : X \to B$ under the action of the cotangent line bundle $\gamma : \Omega^1_B \to B$ of B.*

Proof. Since C is an anti-canonical divisor on V, it follows from adjunction formula that every line on V intersects C transversally in a unique point. The image of C by the contraction $\tau : V \to \mathbb{P}^2$ of any 6-tuple of disjoint lines, $L_1, \ldots, L_6$, on V is therefore a rational cuspidal cubic containing the images $q_i = \tau(L_i)$, $i = 1, \ldots, 6$, in its regular locus. The rational pencil $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ generated by $\tau(C)$ and three times its tangent line T at its singular point lifts to a rational pencil $\overline{f} : V \dashrightarrow \mathbb{P}^1$ whose restriction to X is an $\mathbb{A}^1$-fibration $f : X \to \mathbb{P}^1$ with two degenerate fibers: one irreducible of multiplicity three consisting of the intersection of the proper transform of T with X, and a reduced one consisting of the disjoint union of the curves $L_i \cap X \simeq \mathbb{A}^1$, $i = 1, \ldots, 6$. Choosing homogeneous coordinates $[z_0 : z_1]$ on $\mathbb{P}^1$ in such a way that 0 and ∞ are the respective images of T and C by $\overline{f}$, the same argument as in [3, §4] implies that $f : X \to \mathbb{P}^1$ factors through an étale locally trivial $\mathbb{A}^1$-bundle $\rho : X \to B$. Letting $\gamma : L \to B$ be the line bundle under which $\rho : X \to B$ becomes a principal homogeneous bundle, it follows from the relative cotangent exact sequence

$$0 \to \rho^*\Omega^1_B \to \Omega^1_X \to \Omega^1_{X/B} \simeq \rho^* L^\vee \to 0$$

of ρ that $\det \Omega^1_X \simeq \rho^*(\Omega^1_B \otimes L^\vee)$. Since $\det \Omega^1_X$ is trivial as C is an anti-canonical divisor on V and since $\rho^* : \mathrm{Pic}(B) \to \mathrm{Pic}(X)$ is an isomorphism because $\rho : X \to B$ is a locally trivial $\mathbb{A}^1$-bundle, we conclude that $L \simeq \Omega^1_B$. ☐

Remark 6. By construction of $\delta : B \to \mathbb{P}^1$, we have $\delta^{-1}(0) = 3p_0$ and $\delta^{-1}(\infty) = \sum_{i=1}^{6} \infty_i$. The Picard group $\mathrm{Pic}(B)$ of B is thus isomorphic to $\mathbb{Z}^6$ generated by the classes of the lines bundle $\mathcal{O}_B(p_0)$, $\mathcal{O}_B(\infty_i)$, $i = 1, \ldots, 6$, with the unique relation $\mathcal{O}_B(3p_0) = \mathcal{O}_B(\sum_{i=1}^{6} \infty_i)$. Furthermore, since δ is étale except at p_0 where it has ramification index 3, we deduce from the ramification formula for the morphism $\delta : B \to \mathbb{P}^1$ that the cotangent bundle $\gamma : \Omega^1_B \to B$ of B is isomorphic to

$$\delta^*\Omega_{\mathbb{P}^1} \otimes_{\mathcal{O}_B} \mathcal{O}_B(2p_0) \simeq \delta^*(\mathcal{O}_{\mathbb{P}^1}(-\{0\} - \{\infty\})) \otimes_{\mathcal{O}_B} \mathcal{O}_B(2p_0) \simeq \mathcal{O}_B(-p_0 - \sum_{i=1}^{6} \infty_i).$$

1.2.3. Now we are ready for the second step of the construction, which completes the proof of the theorem. Letting X_i, $i = 1, 2$, be the complements of irreducible plane cuspidal hyperplanes sections $C_i = V_i \cap H_i$ of smooth cubic surfaces $V_i \subset \mathbb{P}^3$, Proposition 5 asserts the existence of principal homogeneous bundles $\rho_i : X_i \to B$ under the action of the cotangent line bundle $\gamma : \Omega^1_B \to B$ of B. The fiber product $W = X_1 \times_B X_2$ inherits via the first and second projections respectively the structure of a principal homogeneous bundle $\pi_i : W \to X_i$ under $\rho_i^*\Omega^1_B$, $i = 1, 2$. Since X_i is affine, the vanishing of $H^1_{\text{ét}}(X_i, \rho_i^*\Omega^1_B)$ implies that these bundles are both trivial, yielding isomorphisms $\rho_1^*\Omega^1_B \simeq W \simeq \rho_2^*\Omega^1_B$. Letting $q : E \to W$ be the pull-back of the dual $(\Omega^1_B)^\vee$ of Ω^1_B by the morphism $\rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 : W \to B$, $\pi_i \circ q : E \to S_i$ is a vector bundle over X_i isomorphic to the direct sum of $\rho_i^*\Omega^1_B$ and

$\rho_i^*(\Omega^1_B)^\vee$:

$$\begin{tikzcd}
X_1\times\mathbb{A}^2 \simeq_{X_1} \rho_1^*\Omega^1_B\oplus\rho_1^*(\Omega^1_B)^\vee \arrow[rr, "\sim"] \arrow[d] & & E \arrow[d, "q"'] & & \rho_2^*\Omega^1_B\oplus\rho_2^*(\Omega^1_B)^\vee \simeq_{X_2} X_1\times\mathbb{A}^2 \arrow[ll, "\sim"'] \arrow[d] \\
\rho_1^*\Omega^1_B \arrow[rr, "\sim"] \arrow[d] & & X_1\times_B X_2 \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & & \rho_2^*\Omega^1_B \arrow[ll, "\sim"'] \arrow[d] \\
X_1 \arrow[drr, "\rho_1"] & & & & X_2 \arrow[dll, "\rho_2"'] \\
& & B & &
\end{tikzcd}$$

So by virtue of [13, Theorem 3.1], E is isomorphic as a vector bundle over X_i to $\det(\rho_i^*\Omega^1_B\oplus\rho_i^*(\Omega^1_B)^\vee)\oplus\mathbb{A}^1_{X_i}\simeq\mathbb{A}^1_{X_i}\oplus\mathbb{A}^1_{X_i}$ providing the desired isomorphisms $X_1\times\mathbb{A}^2\simeq E\simeq X_2\times\mathbb{A}^2$.

Example 7. Let $V\subset\mathbb{P}^3$ be a general smooth cubic surface and let $\Delta\subset V$ be the curve consisting of points p of V at which the projective tangent hyperplane $T_pV\subset\mathbb{P}^3$ of V at p intersects V along a cuspidal cubic. Let $\mathcal{V}=\Delta\times V$ and let $\mathcal{C}\subset\mathcal{V}$ be relatively ample Cartier divisor with respect to $\mathrm{pr}_\Delta:\mathcal{V}\to\Delta$ whose fiber $\mathcal{C}_p$ over every point $p\in\Delta$ is equal to the intersection $C_p=V\cap T_pV$. Since $\mathrm{Aut}(V)$ is trivial, the pairs (V,C_p), $p\in\Delta$, are pairwise non isomorphic, and so $\Theta=\mathrm{pr}_\Delta\mid_{\mathcal{X}}:\mathcal{X}=\mathcal{V}\setminus\mathcal{C}\to\Delta$ is a family of pairwise non isomorphic rigid smooth affine surfaces whose $\mathbb{A}^2$-cylinders are all isomorphic.

Remark 8. The $\mathbb{A}^2$-cylinder $X\times\mathbb{A}^2$ over the complement X of a cuspidal hyperplane section C of a smooth cubic surface V is *flexible in codimension* 1, that is, for every closed point p outside a possible empty closed subset $Z\subset X\times\mathbb{A}^2$ of codimension at least two, the tangent space $T_{X\times\mathbb{A}^2,p}$ of $X\times\mathbb{A}^2$ at p is spanned by tangent vectors to orbits of algebraic $\mathbb{G}_a$-actions on $X\times\mathbb{A}^2$. This can be seen as follows: one first constructs by a similar procedure as in [3, §3.2] a flexible mate S for X, in the form of smooth affine surface flexible in codimension 1 admitting an $\mathbb{A}^1$-fibration $\pi:S\to\mathbb{P}^1$ which factors through a principal homogeneous bundle $\tilde{\pi}:S\to B$ under the action of a certain line bundle $\gamma':L\to B$. The fiber product $S\times_B X$ is then a smooth affine threefold which is simultaneously isomorphic to the total spaces of the line bundles $\tilde{\pi}^*\Omega^1_B$ and ρ^*L over S and X via the first and second projection respectively. Since S is flexible in codimension 1, it follows from [3, Lemma 2.3] that $S\times_B X$ and the total space $F\to S\times_B X$ of the pull-back of $L^\vee$ by the morphism $\tilde{\pi}\circ\mathrm{pr}_S=\rho\circ\mathrm{pr}_X$ are both flexible in codimension 1. By construction, F is a vector bundle of rank 2 over X, isomorphic to $\rho^*(L\oplus L^\vee)$ hence to the trivial vector bundle $X\times\mathbb{A}^2$ by virtue of [13, Theorem 3.1].

References

1. S. S. Abhyankar, P. Eakin and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, Journal of Algebra, 23(2) (1972), 310-342.
2. W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*, Preprint, Warsaw, 1989.
3. A. Dubouloz, *Flexible bundles over rigid affine surfaces*, Comment. Math. Helvetici 90 (2015),121-137.
4. K.-H. Fieseler, *On complex affine surfaces with $\mathbb{C}^+$-action*, Comment. Math. Helv. 69 (1994), 5-27.
5. G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encycl. Math. Sci., 136, Inv. Theory and Alg. Tr. Groups, VII, Springer-Verlag, 2006.
6. T. Fujita, *On zariski problem*, Proceedings of the Japan Academy, Series A, Mathematical Sciences, 55(3), (1979),106-110.
7. N. Gupta, *On the cancellation problem for the affine space $\mathbb{A}^3$ in characteristc p*, Inventiones Mat. 195, 279-288 (2014).
8. M. Hochster, *Nonuniqueness of coefficient rings in a polynomial ring*, Proceedings of the American Mathematical Society, 34(1) (1972),81-82.
9. S. Iitaka and T. Fujita, *Cancellation theorem for algebraic varieties*, Journal of the Faculty of Science, University of Tokyo, Section 1A, Mathematics, 24(1) (1977),123-127.
10. Z. Jelonek, Affine varieties with stably trivial algebraic vector bundles, Proc. Amer. Math. Soc. 138 (2010), 3105-3109.
11. L. Makar-Limanov, *Locally nilpotent derivations, a new ring invariant and applications*, Lecture notes, Bar-Ilan University, 1998. Avail. at at http://www.math.wayne.edu/ lml/
12. M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. 20 (1980), no. 1, 11-42.
13. M. Pavaman Murthy, *Vector Bundles Over Affine Surfaces Birationally Equivalent to a Ruled Surface*, Annals of Maths. vol. 89, No. 2 (1969), 242-253.
14. P. Russell, *Cancellation*, in Automorphisms in Birational and Affine Geometry, Springer Proceedings in Mathematics and Statistics Volume 79 (2014), 495-518.

IMB UMR5584, CNRS, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France.
E-mail address: adrien.dubouloz@u-bourgogne.fr